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Bethe Ansatz derived from the functional relations of the open XXZ chain for new special cases

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Abstract

The transfer matrix of the general integrable open XXZ quantum spin chain obeys certain functional relations at roots of unity. By exploiting these functional relations, we determine the Bethe Ansatz solution for the transfer matrix eigenvalues for the special cases that all but one of the boundary parameters are zero, and the bulk anisotropy parameter is $i\pi/3\,, i\pi/5\,, \ldots$.

1 Introduction

The open XXZ quantum spin chain with general integrable boundary terms [1] is a fundamental integrable model with boundary, which has applications in condensed matter physics, statistical mechanics and string theory. Although this model remains unsolved, the special case of diagonal boundary terms was solved long ago [2, 3, 4], and some progress on the more general case has been achieved recently by two different approaches. One approach, pursued by Cao *et al.* [5] is an adaptation of the generalized algebraic Bethe Ansatz [6, 7] to open chains. Another approach, which was developed in [8]-[11] and which we pursue further here, exploits the functional relations obeyed by the transfer matrix at roots of unity. It is based on fusion [12], the truncation of the fusion hierarchy at roots of unity [13], and the Bazhanov-Reshetikhin [14] solution of RSOS models.

Both approaches lead to a Bethe Ansatz solution for the special case that the boundary parameters obey a certain constraint. Namely, (following the notation of the second reference in [10] where $\alpha_-\,, \beta_-\,, \theta_-$ and $\alpha_+\,, \beta_+\,, \theta_+$ denote the left and right boundary parameters, respectively, and N is the number of spins in the chain),

$$\alpha_- + \beta_- + \alpha_+ + \beta_+ = \pm(\theta_- - \theta_+) + \eta k\,, \tag{1.1}$$

where k is an even integer if N is odd, and is an odd integer if N is even. This solution has been used to derive a nonlinear integral equation for the sine-Gordon model on an interval [15], and has been generalized to other models [16].

Despite these successes, it would be desirable to find the solution for general values of the boundary parameters; i.e., when the constraint (1.1) is not satisfied. In the functional relation approach, the main difficulty lies in recasting the functional relations (which are known [9, 10] for general values of the boundary parameters) as the condition that a certain determinant vanish. In this note we report the solution of this problem (and hence, the Bethe Ansatz expression for the transfer matrix eigenvalues) for the special cases that all but one of the boundary parameters are zero, and the bulk anisotropy has values $\eta = \frac{i\pi}{3}\,, \frac{i\pi}{5}\,, \ldots$. It may be possible to extend this analysis to more general cases.

In Section 2, we briefly review the construction of the transfer matrix and the functional relations which it satisfies at roots of unity. In Section 3 we present our main results; namely, the Bethe Ansatz solution for the transfer matrix eigenvalues when all but one of the boundary parameters vanish. We conclude in Section 4 with a brief discussion of these results. In an Appendix we briefly review the solution [10, 11] for the case that the constraint (1.1) is satisfied, in order to facilitate comparison with the new cases considered here.

2 Transfer matrix and functional relations

The transfer matrix $t(u)$ of the open XXZ chain with general integrable boundary terms is given by [4]

$$t(u) = \mathrm{tr}_0\, K^+_0(u)\ T_0(u)\ K^-_0(u)\ \hat{T}_0(u)\,, \tag{2.1}$$

where $T_0(u)$ and $\hat{T}_0(u)$ are the monodromy matrices

$$T_0(u) = R_{0N}(u) \cdots R_{01}(u)\,, \qquad \hat{T}_0(u) = R_{01}(u) \cdots R_{0N}(u)\,, \tag{2.2}$$

and tr_0 denotes trace over the "auxiliary space" 0. The R matrix is given by

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}\,, \tag{2.3}$$

where η is the bulk anisotropy parameter; and $K^{\mp}(u)$ are 2×2 matrices whose components are given by [1, 17]

$$\begin{aligned} K^-_{11}(u) &= 2\left(\sinh \alpha_- \cosh \beta_- \cosh u + \cosh \alpha_- \sinh \beta_- \sinh u\right) \\ K^-_{22}(u) &= 2\left(\sinh \alpha_- \cosh \beta_- \cosh u - \cosh \alpha_- \sinh \beta_- \sinh u\right) \\ K^-_{12}(u) &= e^{\theta_-} \sinh 2u\,, \qquad K^-_{21}(u) = e^{-\theta_-} \sinh 2u\,, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} K^+_{11}(u) &= -2\left(\sinh \alpha_+ \cosh \beta_+ \cosh(u+\eta) - \cosh \alpha_+ \sinh \beta_+ \sinh(u+\eta)\right) \\ K^+_{22}(u) &= -2\left(\sinh \alpha_+ \cosh \beta_+ \cosh(u+\eta) + \cosh \alpha_+ \sinh \beta_+ \sinh(u+\eta)\right) \\ K^+_{12}(u) &= -e^{\theta_+} \sinh 2(u+\eta)\,, \qquad K^+_{21}(u) = -e^{-\theta_+} \sinh 2(u+\eta)\,, \end{aligned} \tag{2.5}$$

where $\alpha_{\mp}\,, \beta_{\mp}\,, \theta_{\mp}$ are the boundary parameters. [1]

In addition to the fundamental commutativity property

$$[t(u)\,, t(v)] = 0\,, \tag{2.6}$$

the transfer matrix also has $i\pi$ periodicity

$$t(u + i\pi) = t(u)\,, \tag{2.7}$$

[1]Following [10, 11], we use a parametrization of the boundary parameters which differs from that in [1, 17]. Specifically, the matrices $K^{\mp}(u)$ are equal to those appearing in the second reference in [10] divided by the factors $\kappa_{\mp}$, respectively.

crossing symmetry

$$t(-u-\eta) = t(u)\,, \tag{2.8}$$

and the asymptotic behavior

$$t(u) \sim -\cosh(\theta_- - \theta_+)\frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}}\mathbb{I} + \ldots \qquad \text{for} \qquad u \to \infty\,. \tag{2.9}$$

For bulk anisotropy values $\eta = \frac{i\pi}{p+1}$, with $p = 1\,,2\,,\ldots$, the transfer matrix obeys functional relations of order $p+1$ [9, 10]

$$\begin{aligned} & t(u)t(u+\eta)\ldots t(u+p\eta) \\ & - \quad \delta(u-\eta)t(u+\eta)t(u+2\eta)\ldots t(u+(p-1)\eta) \\ & - \quad \delta(u)t(u+2\eta)t(u+3\eta)\ldots t(u+p\eta) \\ & - \quad \delta(u+\eta)t(u)t(u+3\eta)t(u+4\eta)\ldots t(u+p\eta) \\ & - \quad \delta(u+2\eta)t(u)t(u+\eta)t(u+4\eta)\ldots t(u+p\eta) - \ldots \\ & - \quad \delta(u+(p-1)\eta)t(u)t(u+\eta)\ldots t(u+(p-2)\eta) \\ & + \quad \ldots = f(u)\,. \end{aligned} \tag{2.10}$$

For example, for the case $p = 2$, the functional relation is

$$t(u)t(u+\eta)t(u+2\eta) - \delta(u-\eta)t(u+\eta) - \delta(u)t(u+2\eta) - \delta(u+\eta)t(u) = f(u)\,. \tag{2.11}$$

The functions $\delta(u)$ and $f(u)$ are given in terms of the boundary parameters $\alpha_\mp\,,\beta_\mp\,,\theta_\mp$ by

$$\delta(u) = \delta_0(u)\delta_1(u)\,, \qquad f(u) = f_0(u)f_1(u)\,, \tag{2.12}$$

where

$$\delta_0(u) = (\sinh u \sinh(u+2\eta))^{2N}\,\frac{\sinh 2u \sinh(2u+4\eta)}{\sinh(2u+\eta)\sinh(2u+3\eta)}\,, \tag{2.13}$$

$$\begin{aligned} \delta_1(u) &= 2^4 \sinh(u+\eta+\alpha_-)\sinh(u+\eta-\alpha_-)\cosh(u+\eta+\beta_-)\cosh(u+\eta-\beta_-) \\ &\times \sinh(u+\eta+\alpha_+)\sinh(u+\eta-\alpha_+)\cosh(u+\eta+\beta_+)\cosh(u+\eta-\beta_+)\,, \end{aligned} \tag{2.14}$$

and therefore,

$$\delta(u+i\pi) = \delta(u)\,, \qquad \delta(-u-2\eta) = \delta(u)\,. \tag{2.15}$$

For p even,

$$
\begin{aligned}
f_0(u) &= (-1)^{N+1} 2^{-2pN} \sinh^{2N}\left((p+1)u\right) , && (2.16) \\
f_1(u) &= (-1)^{N+1} 2^{3-2p} \Big(\\
&\quad \sinh\left((p+1)\alpha_-\right) \cosh\left((p+1)\beta_-\right) \sinh\left((p+1)\alpha_+\right) \cosh\left((p+1)\beta_+\right) \cosh^2\left((p+1)u\right) \\
&- \cosh\left((p+1)\alpha_-\right) \sinh\left((p+1)\beta_-\right) \cosh\left((p+1)\alpha_+\right) \sinh\left((p+1)\beta_+\right) \sinh^2\left((p+1)u\right) \\
&- (-1)^N \cosh\left((p+1)(\theta_- - \theta_+)\right) \sinh^2\left((p+1)u\right) \cosh^2\left((p+1)u\right) \Big) . && (2.17)
\end{aligned}
$$

For p odd,

$$
\begin{aligned}
f_0(u) &= (-1)^{N+1} 2^{-2pN} \sinh^{2N}\left((p+1)u\right) \tanh^2\left((p+1)u\right) , && (2.18) \\
f_1(u) &= -2^{3-2p} \Big(\\
&\quad \cosh\left((p+1)\alpha_-\right) \cosh\left((p+1)\beta_-\right) \cosh\left((p+1)\alpha_+\right) \cosh\left((p+1)\beta_+\right) \sinh^2\left((p+1)u\right) \\
&- \sinh\left((p+1)\alpha_-\right) \sinh\left((p+1)\beta_-\right) \sinh\left((p+1)\alpha_+\right) \sinh\left((p+1)\beta_+\right) \cosh^2\left((p+1)u\right) \\
&+ (-1)^N \cosh\left((p+1)(\theta_- - \theta_+)\right) \sinh^2\left((p+1)u\right) \cosh^2\left((p+1)u\right) \Big) . && (2.19)
\end{aligned}
$$

Hence, $f(u)$ satisfies

$$
f(u+\eta) = f(u) \,, \qquad f(-u) = f(u) \,. \tag{2.20}
$$

The commutativity property (2.6) implies that the eigenvectors $|\Lambda\rangle$ of the transfer matrix $t(u)$ are independent of the spectral parameter u. Hence, the corresponding eigenvalues $\Lambda(u)$ obey the same functional relations (2.10), as well as the properties (2.7) - (2.9).

3 Bethe Ansatz solution for new special cases

We henceforth restrict to *even* values of p (i.e., bulk anisotropy values $\eta = \frac{i\pi}{3}\,, \frac{i\pi}{5}\,, \ldots$), and consider the various special cases that all but one of the boundary parameters are zero.

3.1 $\alpha_- \neq 0$

For the case that all boundary parameters are zero except for α_- (or, similarly, α_+), we find that the functional relations (2.10) for the transfer matrix eigenvalues can be written as

$$
\det \mathcal{M} = 0 \,, \tag{3.1}
$$

where $\mathcal{M}$ is given by the $(p+1)\times(p+1)$ matrix

$$\mathcal{M} = \begin{pmatrix} \Lambda(u) & -h(u) & 0 & \ldots & 0 & -h(-u+p\eta) \\ -h(-u) & \Lambda(u+p\eta) & -h(u+p\eta) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h(u+p^2\eta) & 0 & 0 & \ldots & -h(-u-p(p-1)\eta) & \Lambda(u+p^2\eta) \end{pmatrix} \tag{3.2}$$

(whose successive rows are obtained by simultaneously shifting $u \mapsto u + p\eta$ and cyclically permuting the columns to the right) provided that there exists a function $h(u)$ which has the properties

$$\begin{aligned} h(u+2i\pi) = h\left(u+2(p+1)\eta\right) &= h(u)\,, &(3.3)\\ h(u+(p+2)\eta)\ h(-u-(p+2)\eta) &= \delta(u)\,, &(3.4)\\ \prod_{j=0}^{p} h(u+2j\eta) + \prod_{j=0}^{p} h(-u-2j\eta) &= f(u)\,. &(3.5)\end{aligned}$$

To solve for $h(u)$, we set

$$h(u) = h_0(u)h_1(u)\,, \tag{3.6}$$

with

$$h_0(u) = (-1)^N \sinh^{2N}(u+\eta)\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\,. \tag{3.7}$$

Noting that

$$\begin{aligned} h_0(u+(p+2)\eta)\ h_0(-u-(p+2)\eta) &= \delta_0(u)\,, \\ \prod_{j=0}^{p} h_0(u+2j\eta) = \prod_{j=0}^{p} h_0(-u-2j\eta) &= f_0(u)\,, \end{aligned} \tag{3.8}$$

where $\delta_0(u)$ and $f_0(u)$ are given by (2.13) and (2.16), respectively, we see that $h_1(u)$ must satisfy

$$\begin{aligned} h_1(u+(p+2)\eta)\ h_1(-u-(p+2)\eta) &= \delta_1(u)\,, &(3.9)\\ \prod_{j=0}^{p} h_1(u+2j\eta) + \prod_{j=0}^{p} h_1(-u-2j\eta) &= f_1(u)\,. &(3.10)\end{aligned}$$

Eliminating $h_1(-u-2j\eta)$ in (3.10) using (3.9), we obtain

$$z(u)^2 - z(u)f_1(u) + \prod_{j=0}^{p} \delta_1\left(u+(2j-1)\eta\right) = 0\,, \tag{3.11}$$

where

$$z(u) = \prod_{j=0}^{p} h_1(u + 2j\eta)\,. \tag{3.12}$$

Solving the quadratic equation (3.11) for $z(u)$, making use of the explicit expressions (2.14) and (2.17) for $\delta_1(u)$ and $f_1(u)$, respectively, we obtain

$$z(u) = 2^{-2(p-1)} \cosh^2\left((p+1)u\right) \sinh\left((p+1)u\right) \left(\sinh\left((p+1)u\right) \pm \sinh\left((p+1)\alpha_-\right)\right)\,. \tag{3.13}$$

Notice that this expression for $z(u)$ has periodicity 2η, which is consistent with (3.12) and the assumed periodicity (3.3). Corresponding solutions of (3.12) for $h_1(u)$ are

$$h_1(u) = -4 \cosh^2 u \sinh u \sinh(u \mp \alpha_-) \frac{\cosh\left(\frac{1}{2}(u \pm \alpha_- + \eta)\right)}{\cosh\left(\frac{1}{2}(u \mp \alpha_- - \eta)\right)}\,. \tag{3.14}$$

In short, a function $h(u)$ which satisfies (3.3) - (3.5) is given by

$$\begin{aligned} h(u) &= (-1)^{N+1} 4 \sinh^{2N}(u+\eta) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \cosh^2 u \sinh u \\ &\times \sinh(u-\alpha_-) \frac{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u-\alpha_- -\eta)\right)}\,. \end{aligned} \tag{3.15}$$

The structure of the matrix $\mathcal{M}$ (3.2) suggests that its null eigenvector has the form $(Q(u)\,, Q(u+p\eta)\,, \ldots\,, Q(u+p^2\eta))$, where $Q(u)$ has the periodicity property

$$Q(u + 2i\pi) = Q(u)\,. \tag{3.16}$$

It follows that the transfer matrix eigenvalues are given by

$$\Lambda(u) = h(u) \frac{Q(u+p\eta)}{Q(u)} + h(-u+p\eta) \frac{Q(u-p\eta)}{Q(u)}\,, \tag{3.17}$$

which evidently has the form of Baxter's TQ relation. We make the Ansatz

$$Q(u) = \prod_{j=1}^{M} \sinh\left(\frac{1}{2}(u-u_j)\right) \sinh\left(\frac{1}{2}(u+u_j-p\eta)\right)\,, \tag{3.18}$$

which has the periodicity (3.16) as well as the crossing property [2]

$$Q(-u+p\eta) = Q(u)\,. \tag{3.19}$$

[2]Note that $\Lambda(u) = \Lambda(-u+p\eta) = \Lambda(-u-\eta)$, where the first equality follows from (3.17) and (3.19), and the second equality follows from the $i\pi$ periodicity of $\Lambda(u)$ (which, however, is not manifest from (3.17).)

The asymptotic behavior (2.9) is consistent with having M (the number of zeros u_j of $Q(u)$) given by

$$M = N + p + 1\,, \tag{3.20}$$

which we have confirmed numerically for small values of N and p. Analyticity of $\Lambda(u)$ implies the Bethe Ansatz equations

$$\frac{h(u_j)}{h(-u_j + p\eta)} = -\frac{Q(u_j - p\eta)}{Q(u_j + p\eta)}\,, \qquad j = 1\,, \ldots\,, M\,. \tag{3.21}$$

To summarize, for the special case that p is even and all boundary parameters are zero except for α_-, the eigenvalues of the transfer matrix (2.1) are given by (3.17), where $h(u)$ is given by (3.15), and $Q(u)$ is given by (3.18),(3.20), with zeros u_j given by (3.21).

We observe that for the special case that we are considering, the corresponding Hamiltonian is *not* of the usual XXZ form. Indeed, $t'(0)$ (the first derivative of the transfer matrix evaluated at $u = 0$) is proportional to σ^x_N. Hence, to obtain a nontrivial integrable Hamiltonian, one must consider the second derivative of the transfer matrix. We find

$$\begin{aligned} t''(0) &= -16 \sinh^{2N-1}\eta \cosh\eta \sinh\alpha_- \Bigg(\left\{ \sigma^x_N\,, \sum_{n=1}^{N-1} H_{n\,,n+1} \right\} \\ &+ (N\cosh\eta + \sinh\eta\tanh\eta)\sigma^x_N + \frac{\sinh\eta}{\sinh\alpha_-}\sigma^x_1\sigma^x_N \Bigg)\,, \end{aligned} \tag{3.22}$$

where $H_{n\,,n+1}$ is given by

$$H_{n\,,n+1} = \frac{1}{2}\left(\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + \cosh\eta\ \sigma^z_n\sigma^z_{n+1} \right)\,. \tag{3.23}$$

3.2 $\beta_- \ne 0$

For the case that all boundary parameters are zero except for β_- (or, similarly, β_+), we find that the functional relations (2.10) for the transfer matrix eigenvalues can again be written in the form (3.1), where now the matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \Lambda(u) & -h(u) & 0 & \ldots & 0 & -h(-u-\eta) \\ -h(-u-(p+1)\eta) & \Lambda(u+p\eta) & -h(u+p\eta) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h(u+p^2\eta) & 0 & 0 & \ldots & -h(-u-(p^2+1)\eta) & \Lambda(u+p^2\eta) \end{pmatrix} \tag{3.24}$$

if $h(u)$ satisfies

$$\begin{aligned} h(u+2i\pi) = h\left(u+2(p+1)\eta\right) &= h(u)\,, & (3.25)\\ h(u+(p+2)\eta)\ h(-u-\eta) &= \delta(u)\,, & (3.26)\\ \prod_{j=0}^{p} h(u+2j\eta) + \prod_{j=0}^{p} h(-u-(2j+1)\eta) &= f(u)\,. & (3.27)\end{aligned}$$

Proceeding similarly to the previous case, we now find

$$h(u) = (-1)^N 4\sinh^{2N}(u+\eta)\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\sinh^2 u\cosh u\left(\cosh u + (-1)^{\frac{p}{2}} i \sinh\beta_-\right)\,. \tag{3.28}$$

The transfer matrix eigenvalues are now given by

$$\Lambda(u) = h(u)\frac{Q(u+p\eta)}{Q(u)} + h(-u-\eta)\frac{Q(u-p\eta)}{Q(u)}\,, \tag{3.29}$$

with

$$Q(u) = \prod_{j=1}^{M}\sinh\left(\frac{1}{2}(u-u_j)\right)\sinh\left(\frac{1}{2}(u+u_j+\eta)\right)\,, \tag{3.30}$$

which satisfies $Q(u+2i\pi) = Q(u)$ and $Q(-u-\eta) = Q(u)$; and

$$M = N + p\,. \tag{3.31}$$

Moreover, the Bethe Ansatz equations for the zeros u_j take the form

$$\frac{h(u_j)}{h(-u_j-\eta)} = -\frac{Q(u_j-p\eta)}{Q(u_j+p\eta)}\,, \qquad j = 1\,,\ldots\,,M\,. \tag{3.32}$$

For this case, $t'(0) = 0$, and

$$t''(0) = -16\cosh\eta\sinh^{2N}\eta\,(\sigma_1^x + \sinh\beta_-\ \sigma_1^z)\,\sigma_N^x\,. \tag{3.33}$$

Higher derivatives yield more complicated expressions.

3.3 $\theta_\mp \neq 0$

For the case that all boundary parameters are zero except for θ_- and θ_+ (quantities of interest depend only on the difference $\theta_- - \theta_+$), we find that the functional relations (2.10) for the

transfer matrix eigenvalues can be written in the form (3.1), where the matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \Lambda(u) & -h^{(2)}(-u-\eta) & 0 & \ldots & 0 & -h^{(1)}(u) \\ -h^{(1)}(u+\eta) & \Lambda(u+\eta) & -h^{(2)}(-u-2\eta) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h^{(2)}(-u-(p+1)\eta) & 0 & 0 & \ldots & -h^{(1)}(u+p\eta) & \Lambda(u+p\eta) \end{pmatrix} \tag{3.34}$$

(whose successive rows are obtained by simultaneously shifting $u \mapsto u+\eta$ and cyclically permuting the columns to the right), if the functions $h^{(1)}(u)$ and $h^{(2)}(u)$ satisfy

$$h^{(k)}(u+i\pi) = h^{(k)}\left(u+(p+1)\eta\right) = h^{(k)}(u)\,, \quad k=1\,,2\,, \tag{3.35}$$

$$h^{(1)}(u+\eta)\; h^{(2)}(-u-\eta) = \delta(u)\,, \tag{3.36}$$

$$\prod_{j=0}^{p} h^{(1)}(u+j\eta) + \prod_{j=0}^{p} h^{(2)}(-u-j\eta) = f(u)\,. \tag{3.37}$$

We find

$$\begin{aligned} h^{(1)}(u) &= (-1)^N e^{\theta_+ - \theta_-} \sinh^{2N}(u+\eta) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh^2 2u\,, \\ h^{(2)}(u) &= (-1)^N e^{\theta_- - \theta_+} \sinh^{2N}(u+\eta) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh^2 2u\,. \end{aligned} \tag{3.38}$$

The transfer matrix eigenvalues are given by

$$\Lambda(u) = h^{(1)}(u) \frac{Q(u-\eta)}{Q(u)} + h^{(2)}(-u-\eta) \frac{Q(u+\eta)}{Q(u)}\,, \tag{3.39}$$

with, for N even,

$$Q(u) = \prod_{j=1}^{2M} \sinh(u-u_j)\,, \tag{3.40}$$

which satisfies $Q(u+i\pi) = Q(u)$; and

$$M = \frac{1}{2}(N+p)\,. \tag{3.41}$$

The Bethe Ansatz equations for the zeros u_j take the form

$$\frac{h^{(1)}(u_j)}{h^{(2)}(-u_j-\eta)} = -\frac{Q(u_j+\eta)}{Q(u_j-\eta)}\,, \qquad j=1\,,\ldots\,,M\,. \tag{3.42}$$

For this case, also $t'(0)=0$, and

$$\begin{aligned} t''(0) &= -16 \cosh\eta \sinh^{2N}\eta \Big(\cosh\theta_- \cosh\theta_+\; \sigma_1^x \sigma_N^x + i \cosh\theta_- \sinh\theta_+\; \sigma_1^x \sigma_N^y \\ &+ i \sinh\theta_- \cosh\theta_+\; \sigma_1^y \sigma_N^x - \sinh\theta_- \sinh\theta_+\; \sigma_1^y \sigma_N^y \Big)\,. \end{aligned} \tag{3.43}$$

4 Discussion

We have checked these solutions numerically for chains of length up to $N = 6$, and have verified that they give the complete set of 2^N eigenvalues. Hence, completeness is achieved more simply than in the case that the constraint (1.1) is satisfied [11].

We emphasize that, in contrast to the solution for the case that the constraint (1.1) is satisfied, these solutions do *not* hold for generic values of the bulk anisotropy. Indeed, these solutions hold only for $\eta = \frac{i\pi}{3}, \frac{i\pi}{5}, \ldots$. Also, while the $Q(u)$ functions have periodicity $i\pi$ for the case that the constraint (1.1) is satisfied and for the case treated in Section 3.3, the $Q(u)$ functions have only $2i\pi$ periodicity for the cases treated in Sections 3.1 and 3.2. (See Eqs. (A.10), (3.40), (3.18) and (3.30), respectively.)

Two key steps in our approach for solving for the function $h(u)$ (which permits the recasting of the functional relations (2.10) as the vanishing of a determinant (3.1)) are solving the quadratic equation (3.11) for $z(u)$, and factoring the result, such as in (3.12). For the special cases solved so far (namely, the case (1.1) considered in [5, 10, 11], and the new cases considered here), the discriminants of the corresponding quadratic equations are perfect squares, and the factorizations can be readily carried out. However, for general values of the boundary parameters, the discriminant is no longer a perfect square; and factoring the result becomes a formidable challenge. Perhaps elliptic functions may prove useful in this regard. [3] We hope to report further on this problem in the future.

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A Appendix

Here we briefly review the solution [10, 11] for the case that the constraint (1.1) is satisfied, in order to facilitate comparison with the new cases considered in text. The matrix $\mathcal{M}$ is

[3] An attempt along this line for the case $p = 1$ was considered in [8].

then given by

$$\mathcal{M} = \begin{pmatrix} \Lambda(u) & -h(-u-\eta) & 0 & \dots & 0 & -h(u) \\ -h(u+\eta) & \Lambda(u+\eta) & -h(-u-2\eta) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h(-u-(p+1)\eta) & 0 & 0 & \dots & -h(u+p\eta) & \Lambda(u+p\eta) \end{pmatrix} \quad \text{(A.1)}$$

where $h(u)$ must satisfy

$$h(u+i\pi) = h\left(u+(p+1)\eta\right) \;=\; h(u)\,, \quad \text{(A.2)}$$

$$h(u+\eta)\; h(-u-\eta) \;=\; \delta(u)\,, \quad \text{(A.3)}$$

$$\prod_{j=0}^{p} h(u+j\eta) + \prod_{j=0}^{p} h(-u-j\eta) \;=\; f(u)\,. \quad \text{(A.4)}$$

A pair of solutions is given by $h(u) = h^{(\pm)}(u) = h_0(u) h_1^{(\pm)}(u)$ with $h_0(u)$ given by (3.7), and $h_1^{(\pm)}(u)$ given by

$$h_1^{(\pm)}(u) = (-1)^{N+1} 4 \sinh(u \pm \alpha_-) \cosh(u \pm \beta_-) \sinh(u \pm \alpha_+) \cosh(u \pm \beta_+)\,, \quad \text{(A.5)}$$

Indeed, $h_0(u)$ satisfies

$$\begin{aligned} h_0(u+\eta)\; h_0(-u-\eta) &= \delta_0(u)\,, \\ \prod_{j=0}^{p} h_0(u+j\eta) = \prod_{j=0}^{p} h_0(-u-j\eta) &= f_0(u)\,, \end{aligned} \quad \text{(A.6)}$$

where $\delta_0(u)$ is given by (2.13), and $f_0(u)$ is given by (2.16) and (2.18) for p even and odd, respectively. Moreover, $h_1^{(\pm)}(u)$ satisfies

$$h_1^{(\pm)}(u+\eta)\; h_1^{(\pm)}(-u-\eta) = \delta_1(u)\,, \quad \text{(A.7)}$$

where $\delta_1(u)$ is given by (2.14); and

$$\begin{aligned} \prod_{j=0}^{p} h_1^{(\pm)}(u+j\eta) + \prod_{j=0}^{p} h_1^{(\pm)}(-u-j\eta) &= f_1(u) - (-1)^{p(N+1)} 2^{1-2p} \sinh^2\left(2(p+1)u)\right) \times \\ &\times \left[(-1)^N \cosh\left((p+1)(\alpha_- + \alpha_+ + \beta_- + \beta_+)\right) + \cosh\left((p+1)(\theta_- - \theta_+)\right)\right]\,, \end{aligned} \quad \text{(A.8)}$$

where $f_1(u)$ is given by (2.17) and (2.19) for p even and odd, respectively. Hence, if the constraint (1.1) is satisfied, then the RHS of (A.8) reduces to $f_1(u)$; hence, all the conditions (A.2)-(A.4) are fulfilled. The corresponding expression for the transfer matrix eigenvalues is given by

$$\Lambda^{(\pm)}(u) = h^{(\pm)}(u) \frac{Q^{(\pm)}(u-\eta)}{Q^{(\pm)}(u)} + h^{(\pm)}(-u-\eta) \frac{Q^{(\pm)}(u+\eta)}{Q^{(\pm)}(u)}\,, \quad \text{(A.9)}$$

with

$$Q^{(\pm)}(u) = \prod_{j=1}^{M^{(\pm)}} \sinh(u - u_j^{(\pm)}) \sinh(u + u_j^{(\pm)} + \eta)\,, \qquad M^{(\pm)} = \frac{1}{2}(N - 1 \pm k)\,, \tag{A.10}$$

and Bethe Ansatz equations

$$\frac{h^{(\pm)}(u_j^{(\pm)})}{h^{(\pm)}(-u_j^{(\pm)} - \eta)} = -\frac{Q^{(\pm)}(u_j^{(\pm)} + \eta)}{Q^{(\pm)}(u_j^{(\pm)} - \eta)}\,, \qquad j = 1\,, \ldots\,, M^{(\pm)}\,. \tag{A.11}$$

References

[1] H.J. de Vega and A. González-Ruiz, J. Phys. *A26* (1993) L519.

[2] M. Gaudin, Phys. Rev. *A4* (1971) 386; *La fonction d'onde de Bethe* (Masson, 1983).

[3] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel, J. Phys. *A20* (1987) 6397.

[4] E.K. Sklyanin, J. Phys. *A21* (1988) 2375.

[5] J. Cao, H.-Q. Lin, K.-J. Shi, Y. Wang, "Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields," `cond-mat/0212163`; Nucl. Phys. *B663* (2003) 487.

[6] R.J. Baxter, Ann. Phys. *76* (1973) 1-24, 25-47, 48-71.

[7] L.D. Faddeev and L.A. Takhtajan, Russ. Math Surv. *34* (1979) 11.

[8] R.I. Nepomechie, J. Phys. *A34* (2001) 9993.

[9] R.I. Nepomechie, Nucl. Phys. *B622* (2002) 615; Addendum, Nucl. Phys. *B631* (2002) 519.

[10] R.I. Nepomechie, J. Stat. Phys. *111* (2003) 1363; J. Phys. *A37* (2004) 433.

[11] R.I. Nepomechie and F. Ravanini, J. Phys. *A36* (2003) 11391; Addendum, J. Phys. *A37* (2004) 1945.

[12] P.P. Kulish and E.K. Sklyanin, *Lecture Notes in Physics*, Vol. 151, (Springer, 1982) 61; P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. *5* (1981) 393; P.P. Kulish and N.Yu. Reshetikhin, J. Sov. Math. *23* (1983) 2435; A.N. Kirillov and N.Yu. Reshetikhin, J. Sov. Math. *35* (1986) 2627; J. Phys. *A20* (1987) 1565; L. Mezincescu and R.I. Nepomechie, J. Phys. *A25* (1992) 2533; Y.-K. Zhou, Nucl. Phys. *B458* (1996) 504.

[13] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, Commun. Math. Phys. *177* (1996) 381; Commun. Math. Phys. *200* (1999) 297; A. Kuniba, K. Sakai and J. Suzuki, Nucl. Phys. *B525* (1998) 597.

[14] V.V. Bazhanov and N.Yu. Reshetikhin, Int. J. Mod. Phys. *A4* (1989) 115.

[15] C. Ahn and R.I. Nepomechie, Nucl. Phys. *B676* (2004) 637; C. Ahn, Z. Bajnok, R.I. Nepomechie, L. Palla and G. Takács, Nucl. Phys. *B714* (2005) 307.

[16] A. Doikou, Nucl. Phys. *B668* (2003) 447; J. de Gier and P. Pyatov, JSTAT *03* (2004) 002; W. Galleas and M.J. Martins, Phys. Lett. *A335* (2005) 167; C.S. Melo, G.A.P. Ribeiro and M.J. Martins, Nucl. Phys. *B711* (2005) 565; W.-L. Yang and Y.-Z. Zhang, JHEP *01* (2005) 021.

[17] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. *A9* (1994) 3841.

ADDENDUM to "Bethe Ansatz derived from the functional relations of the open XXZ chain for new special cases"

In [1] (to which we refer hereafter by I), we obtain Bethe Ansatz solutions for the transfer matrix eigenvalues of the open XXZ chain for the special cases that the bulk anisotropy parameter has values

$$\eta = \frac{i\pi}{p+1}\,, \qquad p = 2\,, 4\,, 6\,, \ldots\,, \tag{1}$$

and *one* of the boundary parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ is arbitrary, and the remaining boundary parameters are zero. Here we show that those results can readily be extended to the cases that any *two* of the boundary parameters $\{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ are arbitrary and the remaining boundary parameters are either η or $i\pi/2$. (We assume that $\theta_- = \theta_+ \equiv \theta$.) For these cases, the corresponding Hamiltonians have the conventional local form (see, e.g., [2])

$$\begin{aligned}\mathcal{H} &= \sum_{n=1}^{N-1} H_{n\,,n+1} + \frac{1}{2}\sinh\eta\Big[\coth\alpha_-\tanh\beta_-\sigma_1^z + \operatorname{csch}\alpha_-\operatorname{sech}\beta_-(\cosh\theta\sigma_1^x + i\sinh\theta\sigma_1^y) \\ &- \coth\alpha_+\tanh\beta_+\sigma_N^z + \operatorname{csch}\alpha_+\operatorname{sech}\beta_+(\cosh\theta\sigma_N^x + i\sinh\theta\sigma_N^y)\Big]\,,\end{aligned} \tag{2}$$

where $H_{n\,,n+1}$ is given by (I3.23). The corresponding energy eigenvalues are related to the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ (I2.1) by

$$E = c_1 \frac{\partial}{\partial u}\Lambda(u)\Big|_{u=0} + c_2\,, \tag{3}$$

where

$$\begin{aligned}c_1 &= -\frac{1}{16\sinh\alpha_-\cosh\beta_-\sinh\alpha_+\cosh\beta_+\sinh^{2N-1}\eta\cosh\eta}\,, \\ c_2 &= -\frac{\sinh^2\eta + N\cosh^2\eta}{2\cosh\eta}\,.\end{aligned} \tag{4}$$

1 $\alpha_-\,, \alpha_+$ arbitrary

For the case that $\alpha_\pm$ are arbitrary and $\beta_\pm = \eta$, we find that

$$\begin{aligned}\sqrt{f_1(u)^2 - 4\prod_{j=0}^{p}\delta_1\left(u + (2j-1)\eta\right)} &= 2^{-2p+3}\cosh^2\left((p+1)u\right)\sinh\left((p+1)u\right) \\ &\times \left[\sinh\left((p+1)\alpha_-\right) - (-1)^N\sinh\left((p+1)\alpha_+\right)\right]\,. \end{aligned} \tag{5}$$

The key point is that the argument of the square root is a perfect square. For definiteness, we henceforth restrict to *even* values of N. It follows that the quantity $z(u)$ appearing in (I3.11) is now given by (cf. (I3.13))

$$\begin{aligned} z(u) &= 2^{-2(p-1)} \cosh^2\left((p+1)u\right) \left[\sinh\left((p+1)u\right) \pm \sinh\left((p+1)\alpha_-\right)\right] \\ &\times \left[\sinh\left((p+1)u\right) \mp \sinh\left((p+1)\alpha_+\right)\right] . \end{aligned} \tag{6}$$

Corresponding solutions of (I3.12) for $h_1(u)$ are (cf. (I3.14))

$$h_1(u) = 4\cosh^2(u-\eta)\sinh(u \mp \alpha_-)\sinh(u \pm \alpha_+) \frac{\cosh\left(\frac{1}{2}(u \pm \alpha_- + \eta)\right)}{\cosh\left(\frac{1}{2}(u \mp \alpha_- - \eta)\right)} \frac{\cosh\left(\frac{1}{2}(u \mp \alpha_+ + \eta)\right)}{\cosh\left(\frac{1}{2}(u \pm \alpha_+ - \eta)\right)} . \tag{7}$$

Hence, for $h(u) = h_0(u) h_1(u)$ we can take (cf. (I3.15))

$$\begin{aligned} h(u) &= 4\sinh^{2N}(u+\eta) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \cosh^2(u-\eta) \\ &\times \sinh(u-\alpha_-)\sinh(u+\alpha_+) \frac{\cosh\left(\frac{1}{2}(u + \alpha_- + \eta)\right)}{\cosh\left(\frac{1}{2}(u - \alpha_- - \eta)\right)} \frac{\cosh\left(\frac{1}{2}(u - \alpha_+ + \eta)\right)}{\cosh\left(\frac{1}{2}(u + \alpha_+ - \eta)\right)} , \end{aligned} \tag{8}$$

which indeed satisfies (I3.3)-(I3.5). The transfer matrix eigenvalues and Bethe Ansatz equations are given by (I3.17), (I3.18), (I3.21), with (cf. (I3.20))

$$M = N + 2p + 1 . \tag{9}$$

2 β_- , β_+ arbitrary

For the case that $\beta_\pm$ are arbitrary and $\alpha_\pm = \eta$, we find that

$$\begin{aligned} \sqrt{f_1(u)^2 - 4\prod_{j=0}^{p} \delta_1\left(u + (2j-1)\eta\right)} &= i 2^{-2p+3} \sinh^2\left((p+1)u\right) \cosh\left((p+1)u\right) \\ &\times \left[\sinh\left((p+1)\beta_-\right) - \sinh\left((p+1)\beta_+\right)\right] , \end{aligned} \tag{10}$$

and therefore

$$\begin{aligned} z(u) &= 2^{-2(p-1)} \sinh^2\left((p+1)u\right) \left[\cosh\left((p+1)u\right) \pm i \sinh\left((p+1)\beta_-\right)\right] \\ &\times \left[\cosh\left((p+1)u\right) \mp i \sinh\left((p+1)\beta_+\right)\right] . \end{aligned} \tag{11}$$

Thus, we take the function $h(u)$ to be (cf. (I3.28))

$$\begin{aligned} h(u) &= 4\sinh^{2N}(u+\eta) \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh^2(u-\eta) \\ &\times \left(\cosh u + i \sinh \beta_-\right)\left(\cosh u - i \sinh \beta_+\right) , \end{aligned} \tag{12}$$

which indeed satisfies (I3.25)-(I3.27). The transfer matrix eigenvalues and Bethe Ansatz equations are given by (I3.29), (I3.30), (I3.32), with (cf. (I3.31))

$$M = N + 2p - 1 \,. \tag{13}$$

3 $\alpha_-\,, \beta_-$ arbitrary

For the case that $\alpha_-\,, \beta_-$ are arbitrary and $\alpha_+ = i\pi/2$, $\beta_+ = \eta$, we find that

$$\begin{aligned} \sqrt{f_1(u)^2 - 4\prod_{j=0}^{p}\delta_1\left(u + (2j-1)\eta\right)} &= 2^{-2p+3}\cosh^2\left((p+1)u\right)\sinh\left((p+1)u\right) \\ &\times \Big[\sinh\left((p+1)\alpha_-\right) + (-1)^{\frac{p}{2}} i\cosh\left((p+1)\beta_-\right)\Big], \end{aligned} \tag{14}$$

and therefore

$$\begin{aligned} z(u) &= 2^{-2(p-1)}\cosh^2\left((p+1)u\right)\left[\sinh\left((p+1)u\right) \pm \sinh\left((p+1)\alpha_-\right)\right] \\ &\times \Big[\sinh\left((p+1)u\right) \pm (-1)^{\frac{p}{2}} i\cosh\left((p+1)\beta_-\right)\Big] \,. \end{aligned} \tag{15}$$

For $h(u)$ we take

$$\begin{aligned} h(u) &= 4\sinh^{2N}(u+\eta)\frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)}\cosh(u-\eta)\cosh u \\ &\times \sinh(u-\alpha_-)\frac{\cosh\left(\frac{1}{2}(u+\alpha_-+\eta)\right)}{\cosh\left(\frac{1}{2}(u-\alpha_--\eta)\right)}\left(\sinh u + i\cosh\beta_-\right) \,, \end{aligned} \tag{16}$$

which satisfies (I3.3)-(I3.5). The transfer matrix eigenvalues and Bethe Ansatz equations are given by (I3.17), (I3.18), (I3.21), with (cf. (I3.20))

$$M = N + p \,. \tag{17}$$

Similar results hold for the case that $\alpha_+\,, \beta_+$ are arbitrary and $\alpha_- = i\pi/2$, $\beta_- = \eta$, etc.

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References

[1] R. Murgan and R.I. Nepomechie, J. Stat. Mech. (JSTAT) *P05007* (2005).

[2] R.I. Nepomechie and F. Ravanini, J. Phys. *A36* (2003) 11391.